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Characterizing 2-Distance Graphs and Solving the Equations $T_2(X) = kP_2$ or $K_m \cup K_n$

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Abstract

Let X be a finite, simple graph with vertex set $V(X)$. The 2-distance graph $T_2(X)$ of X is the graph with the same vertex set as X and two vertices are adjacent if and only if their distance in X is exactly 2. A graph G is a 2-distance graph if there exists a graph X such that $T_2(X) = G$. In this paper, we give three characterizations of 2-distance graphs, and find all graphs X such that $T_2(X) = kP_2$ or $K_m \cup K_n$, where $k \geq 2$ is an integer, P_2 is the path of order 2, and K_m is the complete graph of order $m \geq 1$.

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1 Introduction

By a graph, we mean a finite, simple, but not necessarily connected graph. Let X be a graph with vertex set $V(X)$ and edge set $E(X)$. For each $u \in V(X)$, let $N_X(u)$ be the neighborhood of u in X (that is, the set of all vertices of X that are adjacent to u), and the cardinality of $N_X(u)$ is the *degree* of u in X , which is denoted by $\deg_X(u)$.

The *distance* between two vertices u and v in X , denoted by $\text{dist}_X(u, v)$, is the length of a shortest u - v path in X . For convenience, we set $\text{dist}_X(u, v) = 0$ if and only if $u = v$, and $\text{dist}_X(u, v) = \infty$ if and only if u and v are in different components of X . The number $\text{diam}(X) = \max\{\text{dist}_X(u, v) : u, v \in V(X)\}$ is the *diameter* of X .

The *2-distance graph* of X , denoted by $T_2(X)$, is the graph with $V(T_2(X)) = V(X)$ in which two vertices u and v are adjacent in $T_2(X)$ if and only if $\text{dist}_X(u, v) = 2$. A graph G is a *2-distance graph* if there exists a graph X such that $T_2(X) = G$, where equality refers to graph isomorphism. Furthermore, we say that a graph X_0 is a *solution* to the equation $T_2(X) = G$ if $T_2(X_0) = G$.

The idea of 2-distance graph is a particular case of a general notion of k -distance graph $T_k(X)$, which was first studied by Harary, Hoede, and Kedlacek [5]. They investigated the connectedness of 2-distance graph. In the book by Prisner [6, pp157-159], the dynamics of k -distance operator was explored. Furthermore, Boland, Haynes, and Lawson [3] extended the k -distance operator to a graph invariant they call distance- n domination number. Recently, Azimi and Farrokhi [1] studied all graphs whose 2-distance graphs have maximum degree 2. They also solved the problem of finding all graphs whose 2-distance graphs are paths or cycles.

In this paper, we give three characterizations of 2-distance graphs, and find all graphs X such that $T_2(X) = kP_2$ or $K_m \cup K_n$, where $k \geq 2$ is an integer, P_2 is the path of order 2, and K_m is the complete graph of order $m \geq 1$. We end the paper with some open problems.

Given a graph X , we denote its complement and its maximum degree by X^c and $\Delta(G)$, respectively. Other graph-theoretic terms and notations that are not explicitly defined here can be found in [4].

2 Characterizations of 2-distance graphs

We now present the first of the three characterizations of 2-distance graphs.

Theorem 2.1. *Let G be a graph. The following properties are equivalent:*

- (i) G is a 2-distance graph;
- (ii) for every $v_1v_2 \in E(G)$, there exists $v_3 \in V(G)$ that is not adjacent to both v_1 and v_2 in G ; and
- (iii) $\text{diam}(G^c) \leq 2$.

Proof. (i) \Rightarrow (ii): Suppose $G = T_2(X)$ for some graph X . Let v_1 and v_2 be adjacent vertices in G . Then $\text{dist}_X(v_1, v_2) = 2$. Thus, there exists a vertex v_3 adjacent to both v_1 and v_2 in X . This implies that v_3 is not adjacent to both v_1 and v_2 in G .

(ii) \Rightarrow (iii): We consider two cases.

CASE 1. Suppose G^c is a complete graph. Then $\text{diam}(G^c) = 0$ or 1 .

CASE 2. Suppose there exist two vertices in G^c that are not adjacent, say v_1 and v_2 . It implies that v_1 and v_2 are adjacent in G , and, by assumption, there exists a vertex v_3 not adjacent to both of them in G . It follows that v_1 and v_2 are both adjacent to v_3 in G^c , and so $\text{dist}_{G^c}(v_1, v_2) = 2$. Therefore, we have $\text{diam}(G^c) \leq 2$.

(iii) \Rightarrow (i): If $\text{diam}(G^c) = 0$ or 1 , then G^c is a complete graph, and so G is the empty graph. This implies that $T_2(G^c) = G$. On the other hand, if $\text{diam}(G^c) = 2$, then the vertices that are adjacent in $T_2(G^c)$ are exactly those vertices that are not adjacent in G^c . Thus, we get $T_2(G^c) = (G^c)^c = G$. \square

The following corollaries are quick consequences of the preceding theorem.

Corollary 2.2. *Every disconnected graph is a 2-distance graph.*

Corollary 2.3. *Let G be a 2-distance graph. Then G^c is a solution to the equation $T_2(X) = G$. Moreover, if X_0 is a solution to $T_2(X) = G$, then $E(X_0) \subseteq E(G^c)$.*

Our second characterization of 2-distance graphs utilizes a result by Bloom, Kennedy, and Quintas [2] that characterizes graphs with diameter 2.

Let u be a vertex of a graph X . A *star centered at u* is a subgraph of X consisting of edges that have u as a common vertex. Let uv be an edge of X . A *double-star on uv* , or simply a *double-star*, is a maximal tree in X that is the union of stars centered at u or v such that both stars contain uv .

Let Y be a subgraph of a graph X . We say that Y *spans* X if $V(Y) = V(X)$.

Lemma 2.4. [2, Theorem 1] *A graph G has diameter two if and only if G^c is not empty and G^c is not spanned by a double-star.*

Theorem 2.5. *A graph G is a 2-distance graph if and only if G is not spanned by a double-star.*

Proof. Assume that G is a 2-distance graph. By Theorem 2.1, it follows that $\text{diam}(G^c) \leq 2$. If $\text{diam}(G^c) = 0$ or 1, then G is empty, and so G is not spanned by a double-star. If $\text{diam}(G^c) = 2$, then, by Lemma 2.4, G is not spanned by a double-star.

Assume that G is not spanned by a double-star. If G is not empty, then, by Lemma 2.4, we have $\text{diam}(G^c) = 2$, and so G is a 2-distance graph by Theorem 2.1. If G is empty, then G^c is a complete graph, and so $\text{diam}(G^c) = 0$ or 1, and, by Theorem 2.1 again, G is a 2-distance graph. \square

Finally, we give the third characterization of 2-distance graphs, which emphasizes the degrees of the vertices.

We first mention the following lemma, whose proof is immediate.

Lemma 2.6. *Let G be a graph with at least one edge, and let $ab \in E(G)$. If $\deg_G(a) + \deg_G(b) < |V(G)|$, then $V(G) \setminus (N_G(a) \cup N_G(b)) \neq \emptyset$.*

Theorem 2.7. *A graph G is a 2-distance graph if and only if, for each $v \in V(G)$ with $\deg_G(v) \geq \frac{1}{2}|V(G)|$ and for each $a \in N_G(v)$, there exists $b \in V(G)$ such that $b \notin N_G(v) \cup N_G(a)$.*

Proof. Without loss of generality, we assume that G has at least one edge.

Suppose G is a 2-distance graph. Then the conclusion follows at once from Theorem 2.1.

Conversely, suppose that, for each $v \in V(G)$ with $\deg_G(v) \geq \frac{1}{2}|V(G)|$ and for each $a \in N_G(v)$, there exists a vertex that is not adjacent to both v and a . Let $x, y \in V(G)$. If $xy \notin E(G)$, then $\text{dist}_{G^c}(x, y) = 1$. Now, suppose $xy \in E(G)$. We consider two cases.

CASE 1. If $\deg_G(x) \geq \frac{1}{2}|V(G)|$ or $\deg_G(y) \geq \frac{1}{2}|V(G)|$, then $\text{dist}_{G^c}(x, y) = 2$ by assumption.

CASE 2. If $\deg_G(x) < \frac{1}{2}|V(G)|$ and $\deg_G(y) < \frac{1}{2}|V(G)|$, then $\deg_G(x) + \deg_G(y) < |V(G)|$. By Lemma 2.6, we have $\text{dist}_{G^c}(x, y) = 2$.

Thus, we have $\text{diam}(G^c) = 2$, and, by Theorem 2.1, G is a 2-distance graph. \square

Drawing ideas from the proof of the preceding theorem, the following corollaries follow immediately.

Corollary 2.8. *Let G be a graph with $\Delta(G) < \frac{1}{2}|V(G)|$. Then G is a 2-distance graph.*

Corollary 2.9. *Let G be a graph with at least one edge. Then G is a 2-distance graph if and only if $\text{diam}(G^c) = 2$.*

3 Solutions to the Equation $T_2(X) = kP_2$, $k \geq 2$

A quick application of Theorem 2.1 shows that the path P_2 is not a 2-distance graph. However, the union of at least two paths P_2 is a 2-distance graph. In this section, we find all graphs that satisfy the equation $T_2(X) = kP_2$ for $k \geq 2$.

Let K_{2n} be the complete graph of order $2n$, $n \geq 2$, with

$$V(K_{2n}) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}.$$

A *cocktail party graph* of order $2n$, denoted by CP_{2n} , is the graph with

$$V(CP_{2n}) = V(K_{2n}) \quad \text{and} \quad E(CP_{2n}) = E(K_{2n}) \setminus \{u_i v_i : i = 1, 2, \dots, n\}.$$

It is not difficult to check that $T_2(P_4) = 2P_2$, where P_4 is the path of order 4, and $T_2(CP_{2n}) = nP_2$. In general, if $X = pP_4 \cup qCP_{2n}$, where $p \geq 0$, $q \geq 0$, and $n \geq 2$ are integers, then $T_2(X) = (2p + qn)P_2$.

Theorem 3.1. *Let X be a connected graph such that $T_2(X) = kP_2$ for some integer $k \geq 2$. Then*

$$X = \begin{cases} P_4 \text{ or } CP_4 & \text{if } k = 2 \\ CP_{2k} & \text{if } k \geq 3. \end{cases}$$

Proof. As observed in the previous paragraph, we know that $T_2(P_4) = 2P_2$ and $T_2(CP_{2k}) = kP_2$ for any $k \geq 2$.

For $k = 2$, it is not difficult to determine that the only possible solutions are P_4 and CP_4 .

Suppose $k \geq 3$, and let $V(kP_2) = \{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_k\}$ and $E(kP_2) = \{u_i v_i : i = 1, 2, \dots, k\}$. We prove that $X = CP_{2k}$ is the only connected graph that satisfies the property.

For convenience, we only show here that u_1 must be adjacent to u_2 in X , and argue similarly that u_1 must also be adjacent to all other vertices (except v_1).

Suppose, on the contrary, that $u_1 u_2 \notin E(X)$. Then $\text{dist}_X(u_1, u_2) \geq 3$. Because u_1 is the only vertex in X that is of distance 2 from v_1 , we have $\text{dist}_X(u_1, u_2) = 3$. That is, for some vertex v , there is an induced path of the form $u_1 v v_1 u_2$. Since v_2 is the only vertex that is of distance 2 from u_2 , we must have $v = v_2$. Because X is connected and $|V(X)| \geq 6$, we can find another vertex w that is adjacent to one but not all of the vertices u_1, v_2, v_1 , and u_2 . However, in such scenario, one of u_1, v_2, v_1 , and u_2 will be of distance 2 from w , which is a contradiction. Thus, u_1 is adjacent to u_2 in X .

In general, it can be shown in a similar fashion that every vertex u_i (respectively, v_i) is adjacent to all other vertices in X except v_i (respectively, u_i), which implies that $X = CP_{2k}$. \square

The following corollary, whose proof follows from the preceding theorem by separately considering each component of a disconnected graph as a connected graph, enumerates all solutions to the equation $T_2(X) = kP_2$ for $k \geq 2$.

Corollary 3.2. *Let k be an integer greater than or equal to 2. Then the solutions to the equation $T_2(X) = kP_2$ are $X = pP_4 \cup qCP_{2n}$ for all integers $p \geq 0$, $q \geq 0$, and $n \geq 2$ that satisfy the Diophantine equation $2p + qn = k$.*

4 Solutions to the Equation $T_2(G) = K_m \cup K_n$

We know from Theorem 2.1 that $K_m \cup K_n$ is a 2-distance graph for any positive integers m and n . In this section, we solve the equation $T_2(X) = K_m \cup K_n$.

A graph is said to be *bipartite* if its vertex set can be partitioned into two subsets A and B such that each edge has one endvertex in A and the other in B . If every vertex in A is adjacent to every vertex in B , then the graph is a *complete bipartite graph*, which is denoted by $K_{m,n}$, where $|A| = m$ and $|B| = n$.

We observe that being bipartite is a hereditary property; that is, if G is bipartite, then all spanning subgraphs of G are also bipartite, and the sizes of the partitions are also inherited.

The following lemma can be shown easily.

Lemma 4.1. *If $X = K_{1,1}$ or $X = K_{1,1}^c$, then $T_2(X) = K_1 \cup K_1$.*

Lemma 4.2. *Let X be a bipartite graph with partitions of sizes m and n . If $2 \leq \text{diam}(X) \leq 3$, then $T_2(X) = K_m \cup K_n$.*

Proof. When $\text{diam}(X) = 2$, it is not difficult to see that $X = K_{m,n}$, which immediately implies that $T_2(X) = K_m \cup K_n$.

Suppose that $\text{diam}(X) = 3$, and let the partitions of $V(X)$ be A and B with $|A| = m > 1$ and $|B| = n > 1$. Let x and y be distinct vertices of X , and we consider two cases.

CASE 1. If $x, y \in A$ or $x, y \in B$, then $\text{dist}_X(x, y)$ must be of even parity. Since $\text{diam}(X) = 3$, we must have $\text{dist}_X(x, y) = 2$, and so $xy \in T_2(X)$.

CASE 2. If $x \in A$ and $y \in B$ (or $y \in A$ and $x \in B$), then $\text{dist}_X(x, y)$ must be of odd parity, and so $xy \notin T_2(X)$.

Therefore, it follows that if $\text{diam}(X) = 3$, then $T_2(X) = K_m \cup K_n$. \square

Lemma 4.3. *Let m and n be positive integers with at least one of them greater than 1. If $T_2(X) = K_m \cup K_n$, then X is a nonempty spanning subgraph*

of $K_{m,n}$ (that is, X is a nonempty bipartite graph with partitions of sizes m and n).

Proof. By definition, the vertices of K_m are not adjacent to each other in X . The same is true for the vertices of K_n . The vertices of K_m and K_n give rise to a partitioning of $V(X)$ into two sets of sizes m and n , and any edge joins one vertex in one set and another vertex in the other set. Since one of m and n is greater than 1, $K_m \cup K_n$ is nonempty, and so X is also nonempty. Thus, X is a nonempty bipartite graph with partitions of sizes m and n . \square

Theorem 4.4. *Let m and n be positive integers. Then*

- (i) $T_2(X) = K_1 \cup K_1$ if and only if $X = K_{1,1}$ or $X = K_{1,1}^c$; and
- (ii) $T_2(X) = K_m \cup K_n$, where $m > 1$ or $n > 1$, if and only if X is a spanning subgraph of $K_{m,n}$ such that $2 \leq \text{diam}(X) \leq 3$.

Proof. The proof of (i) is straightforward with Lemma 4.1. We now prove (ii).

If X is a spanning subgraph of $K_{m,n}$ such that $2 \leq \text{diam}(X) \leq 3$, then, by Lemma 4.2, $T_2(X) = K_m \cup K_n$.

Suppose that $T_2(X) = K_m \cup K_n$, where $m > 1$ or $n > 1$. By Corollary 2.3 and Lemma 4.3, we know that $(K_m \cup K_n)^c = K_{m,n}$ is a solution to the given equation, and X is a spanning (bipartite) subgraph of $K_{m,n}$ with at least one edge. Since $\text{diam}(K_{m,n}) = 2$, it follows that $\text{diam}(X) \geq 2$.

Suppose $\text{diam}(X) \geq 4$. Then there exist two vertices x and y such that $\text{dist}_X(x, y) = 4$. This implies that x and y belong to one partition of $V(X)$, but they are not of distance 2 in X , which contradicts the fact that they are adjacent in $T_2(X)$ since $T_2(X) = K_m \cup K_n$.

Therefore, we have $\text{diam}(X) \leq 3$, which completes the proof of (ii). \square

The following corollaries are quick consequences of the preceding theorem.

Corollary 4.5. *Let X_0 be a connected bipartite graph with partitions A and B of sizes m and n , respectively. If there exist a vertex in A of degree n and a vertex in B of degree m , then $T_2(X_0) = K_m \cup K_n$. Moreover, any solution to the equation $T_2(X) = K_m \cup K_n$ is a subgraph of a graph having the same property as that of X_0 .*

Corollary 4.6. *Let $n \geq 2$ be a positive integer. Then $T_2(X) = K_1 \cup K_n$ if and only if $X = (K_1 \cup K_n)^c$.*

References

- [1] A. Azimi and M. Farrokhi D.G., Simple graphs whose 2-distance graphs are paths or cycles, *Le Matematiche* **69**(2) (2014) 183-191.
- [2] G.S. Bloom, J.W. Kennedy, and L.V. Quintas, A characterization of graphs of diameter two, *The American Mathematical Monthly* **95**(1) (1988) 37-38.
- [3] J.W. Boland, T.W. Haynes, and L.M. Lawson, Domination from a distance, *Congressus Numerantium* **103** (1994) 89-96.
- [4] J.A. Bondy and U.S.R. Murty, *Graph Theory*, Springer, 2008.
- [5] F. Harary, C. Hoede, and D. Kedlacek, Graph-valued functions related to step graphs, *Journal of Combinatorics, Information, and System Sciences* **7** (1982) 231-246.
- [6] E. Prisner, *Graph Dynamics*, Longman House, 1995.